Solving Differential Equations in Finance and Economics
The Integrating Factor Technique - Part I

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A differential equation is a mathematical equation that relates some function with its derivatives. The solution to a differential equation is not a number but is rather another equation. Differential equations are widely used in finance and economics as these equations can be used to describe how asset prices change over time. Once we define how asset prices change via a differential equation then the solution to that equation is the equation for asset price. For example the Black Scholes model for call option price is the solution to a differential equation that defines a hedge portfolio (consisting of the stock and a risk-free bond) where the change in value of that portfolio over time perfectly offsets any change in value of the asset being hedged (the option on that stock) such that being long the hedge portfolio and short the option is risk-free (i.e. all of the risk of being short the option has been hedged).

In this white paper we will use the Integrating Factor Technique to solve a rather simple differential equation in Part I and a more complex differential equation in Part II. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

We are tasked with tasked with defining the equation for asset price of a bond with the following parameters...

Symbol	Value	Description
B_0	1000.00	Bond price at time zero (in dollars)
α	6.00%	Bond yield to maturity (continuous time)
ϕ	4.00%	Bond coupon payment rate (continuous time)
λ	5.00%	Probability of default (i.e. the hazard rate)

Question 1: What is the equation for bond principal balance as a function of time?

Question 2: What is expected bond principal balance at the end of year 5? (assume that the bond maturity date is greater than five years from time zero)

Defining the Differential Equation

We will define the variable α to be the bond's yield to maturity, the variable ϕ to be the bond's coupon rate, the variable λ to be the probability of bond default over the time interval $[t-\delta t, t]$, the variable t to be time in years, and the variable δt to be an infinitesimally small change in time. If we define the variable B_t to be expected bond principal balance at time t then the equation for the expected change in bond principal balance over the time interval $[t, t+\delta t]$ is...

$$\delta B_t = \left(\alpha - \phi\right) B_t \, \delta t - \lambda \, B_t \, \delta t \tag{1}$$

In Equation (1) above bond principal increases with the accrual of interest (via the yield to maturity) and decreases with coupon payments (disbursement of interest accrued) and defaults (loss of principal). Note that we can rewrite that equation as the following differential equation...

$$\frac{\delta B_t}{\delta t} = \left(\alpha - \phi\right) B_t - \lambda \, B_t = \left(\alpha - \phi - \lambda\right) B_t \tag{2}$$

If we move all terms to the left hand side of the equation then Equation (2) becomes...

$$\frac{\delta B_t}{\delta t} - \bigg(\alpha - \phi - \lambda\bigg) B_t = 0 \tag{3}$$

The solution to the differential equation above (Equation (3)) is the equation for B_t, which is bond principal balance at time t.

Solving the Differential Equation

To solve Equation (3) above we will use the Integrating Factor Technique for solving differential equations. We will start by defining the function I_s to be the intergrating factor whose equation is...

$$I_s = \text{Exp}\bigg\{ - \bigg(\alpha - \phi - \lambda\bigg) s \bigg\} \text{ ...where... } \frac{\delta I_s}{\delta s} = -\bigg(\alpha - \phi - \lambda\bigg) \text{Exp}\bigg\{ - \bigg(\alpha - \phi - \lambda\bigg) s \bigg\} \tag{4}$$

Using the definition of the integrating factor in Equation (4) above the equation for the derivative of the product of the integrating factor and bond principal balance with respect to time is...

$$\begin{aligned} \frac{\delta\, I_s B_s}{\delta s} &= \frac{\delta I_s}{\delta s} B_s + \frac{\delta B_s}{\delta s} I_s \\ &= -\bigg(\alpha - \phi - \lambda\bigg) \text{Exp}\bigg\{ - \bigg(\alpha - \phi - \lambda\bigg) s \bigg\} B_s + \frac{\delta B_s}{\delta s} \text{Exp}\bigg\{ - \bigg(\alpha - \phi - \lambda\bigg) s \bigg\} \\ &= \text{Exp}\bigg\{ - \bigg(\alpha - \phi - \lambda\bigg) s \bigg\} \bigg[\frac{\delta B_s}{\delta s} - \bigg(\alpha - \phi - \lambda\bigg) B_s \bigg] \end{aligned} \tag{5}$$

If we multiply Equation (3) by the integrating factor as defined by Equation (4) above then the equation for that product is...

$$\text{Exp}\bigg\{ - \bigg(\alpha - \phi - \lambda\bigg) s \bigg\} \bigg[\frac{\delta\, B_s}{\delta s} - \bigg(\alpha - \phi - \lambda\bigg) B_s \bigg] = 0 \tag{6}$$

Using Equation (5) above we can rewrite Equation (6) as...

$$\frac{\delta\, I_s B_s}{\delta s} = 0 \tag{7}$$

If we integrate both sides of Equation (7) above then that equation becomes...

$$\int_0^t \frac{\delta\, I_s B_s}{\delta s}\, \delta s = \int_0^t 0\, \delta s \text{ ...becomes... } I_s B_s \bigg[_{s=0}^{s=t} = 0 \text{ ...becomes... } I_t B_t = I_0 B_0 \tag{8}$$

The solution to Equation (8) above, which is the equation for bond principal balance at time t, is...

$$\begin{aligned} \text{Exp}\bigg\{ - \bigg(\alpha - \phi - \lambda\bigg) \times t \bigg\} B_t &= \text{Exp}\bigg\{ - \bigg(\alpha - \phi - \lambda\bigg) \times 0 \bigg\} B_0 \\ \text{Exp}\bigg\{ - \bigg(\alpha - \phi - \lambda\bigg) t \bigg\} B_t &= B_0 \\ B_t &= B_0 \,\text{Exp}\bigg\{ \bigg(\alpha - \phi - \lambda\bigg) t \bigg\} \end{aligned} \tag{9}$$

Our Hypothetical Problem Solutions

The solutions to our hypothetial problem above are...

Question 1: What is the equation for bond principal balance as a function of time?

Answer: The equation for bond principal balance at time t is Equation (9) above.

Question 2: What is expected bond principal balance at the end of year 5?

Answer: Using Equation (9) above and the bond parameters in the table above the expected bond principal balance at the end of year 5 is...

$$B_5 = \$1000 \times \text{Exp}\left\{ \left(0.0600 - 0.0400 - 0.0500 \right) \times 5 \right\} = \$860.71 \tag{10}$$